On smooth surfaces in projective four-space lying on quartic hypersurfaces with isolated singularities.

Ph. Ellia * - D. Franco ** [1]

Dipartimento di Matematica, Università di Ferrara
via Machiavelli 35 - 44100 Ferrara, Italy
* e-mail: phe@dns.unife.it
** e-mail: frv@dns.unife.it

Dedicated to Robin Hartshorne in occasion of his 60th birthday.

Introduction

In the classification of smooth codimension two subvarieties of $\mathbf{P}^n$, surfaces in $\mathbf{P}^4$ (resp. threefolds in $\mathbf{P}^5$) seem to lie between two extremal situations: every curve can be embedded in $\mathbf{P}^3$ while, according to Hartshorne's conjecture, every smooth, codimension two subvariety of $\mathbf{P}^n$, $n \geq 6$, should be a complete intersection.

As well known, not every surface can be embedded in $\mathbf{P}^4$ and, for example, the degrees of smooth rational surfaces in $\mathbf{P}^4$ are bounded (as conjectured by Hartshorne and Lichtenbaum some years ago); more precisely if $S \subset \mathbf{P}^4$ is a smooth surface not of general type, then $deg(S) \leq 46$ ([C]). By the way, it is believed that this result is not optimal and it is conjectured that the sharp bound should be $deg(S) \leq 15$.

It follows from general facts that surfaces of non general type usually lie on hypersurfaces of low degree. So it seems natural to approach this problem by classifying surfaces on hypersurfaces of low degree (a question of independant interest). For hypercubics this has been done by Koelblen ([K]). In this paper we consider surfaces on hyperquartics with isolated singularities, and we prove:

Theorem. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree d lying on a quartic hypersurface with isolated singularities.*
(i) If $p_g = 0$ (so in particular if S is rational), then $d \leq 23$.
(ii) If $h^0(\omega_S(-1)) = 0$ (so in particular if S is of non general type), then $d \leq 27$.

In a few words, our proof (inspired by [A]) goes as follows. On one hand it is not hard, under the assumption $h^0(\omega_S(-1)) = 0$ (resp. $p_g = 0$), to get a

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lower bound $h^2(\mathcal{I}_S(k)) \geq f(k)$ (see Cor. 7, here $d = 4k + r, 0 \leq r \leq 3$). On the other hand, we also have an upper bound: $h^2(\mathcal{I}_S(k)) \leq g(k)$ (see Cor. 3, Remark 3.1), but $g(k)$ depends on cohomological invariants of C, the general hyperplane section of S. We would like to derive a contradiction from the inequality $f(k) \leq g(k)$. In order to apply this naive plan we must have precise informations on the cohomological invariants of C. For this we notice that, by the "Jacobi's formula", $\pi = g(C)$ is not too far from $G(d, 4)$, the maximal genus of a curve of degree d on a quartic surface (see Lemma 8). Here the assumption that the hyperquartic has only isolated singularities is crucial. Then we proceed to a detailed study of the cohomological invariants of curves with genus close to $G(d, 4)$ (Lemma 9, Propositions 12, 14, 16, 18).

It seems worthwhile to stress that the assumptions we really need involve the degrees of the minimal generators of $H^0_*(\omega_S)$, so that our theorem apply also to some surfaces of general type.

Generalities

Lemma 1. *Let $S \subset \mathbf{P}^4$ be a smooth surface and let C denote its general hyperplane section.*

Set $c := max\{k/h^1(\mathcal{I}_C(k)) \neq 0\}$, $v := max\{k/h^2(\mathcal{I}_S(k)) \neq 0\}$.

We have $h^2(\mathcal{I}_S(t)) \leq \sum_{m \geq t+1}^{v+1} h^1(\mathcal{I}_C(m))$ (in particular $h^2(\mathcal{I}_S(m)) = 0$ if $m \geq c$).

Proof. From the exact sequence

$$0 \to \mathcal{I}_S(t) \to \mathcal{I}_S(t+1) \to \mathcal{I}_C(t+1) \to 0$$

it follows that: $h^2(\mathcal{I}_S(t)) \leq h^1(\mathcal{I}_C(t+1)) + h^2(\mathcal{I}_S(t+1))$, we argue by descending induction starting from $t = v$. ∎

The numerical character of a set of points in the plane.

To every zero-dimensional subscheme $\Gamma \subset \mathbf{P}^2$ there is associated a sequence of integers $\chi(\Gamma) = (n_0, ..., n_{\sigma-1})$, called the numerical character of Γ, which encodes the Hilbert function of Γ. We recall the basic properties of the numerical character:

(i) $\sigma = min\{k/h^0(\mathcal{I}_\Gamma(k)) \neq 0)\}$

(ii) $\sum_{i=o}^{\sigma-1} (n_i - i) = d$

(iii) $h^1(\mathcal{I}_\Gamma(n)) = h_\chi(n) := \sum_{i=0}^{\sigma-1} [(n_i - n - 1)_+ - (i - n - 1)_+]$ (where $(x)_+ = max\{x, 0\}$).

The genus of a numerical character is: $g(\chi) := \sum_{m\geq 1} h_\chi(m)$.

If $C \subset \mathbf{P}^3$ is an integral curve, its numerical character, $\chi(C)$, is the numerical character of its general plane section.

The numerical character of an integral curve is connected: $n_i - n_{i+1} \leq 1, 0 \leq i \leq \sigma - 2$ (cf [GP]). Moreover $p_a(C) \leq g(\chi(C))$, with equality if and only if C is a.C.M. (arithmetically Cohen-Macaulay).

Let $\rho_m : H^0(\mathcal{I}_C(m)) \to H^0(\mathcal{I}_\Gamma(m))$ denote the natural map of restriction ($\Gamma = C \cap H$ is the general plane section of C). Set $R_m := coker(\rho_m)$ so that we have an exact sequence of modules of finite length:

$$0 \to R \to M(-1) \to M \to Q \to 0$$

where $M = H^1_*(\mathcal{I}_C)$ and where $Q_m = ker(H^1(\mathcal{I}_\Gamma(m)) \to H^2(\mathcal{I}_C(m-1)))$. Finally set $r_m = dim(R_m)$ and define q_m similarly.

Lemma 2. *With notations as above:*
(i) $\sum_{m\geq 1} r_m = \sum_{m\geq 1} q_m = g(\chi(C)) - g(C)$.
(ii) If $\mathbf{I}(\Gamma)$ *is generated in degrees* $\leq t_0$ *and if* ρ_m *is surjective for some* $m \geq t_0$*, then* ρ_t *is surjective for* $t \geq m$.
(iii) $\forall m$: $h^1(\mathcal{I}_C(m)) \leq \sum_{t\geq m+1} r_t \leq g(\chi(C)) - g(C)$.

Proof. (i) Follows from $g(\chi(C)) = \sum_{m\geq 1} h^1(\mathcal{I}_\Gamma(m))$ and the exact sequences, for $m \geq 1$:

$$0 \to Q_m \to H^1(\mathcal{I}_\Gamma(m)) \to H^2(\mathcal{I}_C(m-1)) \to H^2(\mathcal{I}_C(m)) \to 0$$

(ii) Clear.
(iii) By descending induction. Consider the exact sequence:

$$0 \to R \to M(-1) \to M \to Q \to 0$$

at level $c+1$ it yields $r_{c+1} = h^1(\mathcal{I}_C(c))$. Then at level m it gives:
$h^1(\mathcal{I}_C(m-1)) \leq r_m + h^1(\mathcal{I}_C(m))$. ∎

Corollary 3. *Let* $S \subset \mathbf{P}^4$ *be a smooth surface and let* C *denote its general hyperplane section. Assume* $c > e - 1$*, then:*
$h^2(\mathcal{I}_S(t)) \leq [(c-t).(g(\chi(C)) - g(C))]_+$.

Proof. By Lemma 1: $h^2(\mathcal{I}_S(t)) \leq \sum_{m=t+1}^{c} h^1(\mathcal{I}_C(m))$. Using Lemma 2(iii):

$$h^2(\mathcal{I}_S(t)) \leq \sum_{m\geq t+2} r_m + \sum_{m\geq t+3} r_m + ... + \sum_{m\geq c} r_m + r_{c+1}$$

These are $c-t$ terms and each one is bounded by $\sum_{m\geq 1} r_m = g(\chi(C)) - g(C)$ (see Lemma 2(i)). ∎

Remark 3.1. *The bound of the corollary is very rough, we can improve it, for example, as follows:*

$$h^2(\mathcal{I}_S(t)) \leq [(c-t).(g(\chi(C)) - g(C)) - (c-t).\sum_{m\leq t+1} r_m]_+$$

Lemma 4. *Let C be a smooth connected curve of degree d in $\mathbf{P}^3$ lying on a smooth surface of degree s. Then: $c \leq d + e(1-s) + s^2 - 4s$.*

Proof. See [E2], lemme VI.3. ∎

Lemma 5. *Let $S \subset \mathbf{P}^4$ be a smooth surface with $h^0(\omega_S(-1)) = 0$, then:*
(i) $p_g \leq \pi - \frac{d}{2}$
(ii) If C is linearly normal (for instance if $q = 0$ and $d > 4$) then: $p_g \leq \pi - d + 3$.

Proof. From the assumption $h^0(\omega_S(-1)) = 0$ and the exact sequence:

$$0 \to \omega_S(-1) \to \omega_S \to \omega_C(-1) \to 0$$

we get: $p_g \leq h^0(\omega_C(-1))$.
(i) By Clifford's theorem (of course $\omega_C(-1)$ is special), $h^0(\omega_C(-1)) \leq \pi - \frac{d}{2}$.
(ii) Use $h^0(\omega_C(-1)) = h^1(\mathcal{O}_C(1))$. ∎

Surfaces on hyperquartics with isolated singularities.

Notation: We denote by $G(d,s)$ the maximal genus of a curve of degree d in $\mathbf{P}^3$ not lying in a surface of degree $< s$. If $d > s(s-1)$ then $G(d,s) = 1 + \frac{d(d+s^2-4s)}{2s} - \frac{r(s-1)(s-r)}{2s}$, where $d + r = 0$ (mod s), $0 \leq r < s$.

Lemma 6. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d > 16$ lying on an irreducible hypersurface of degree four. Set $d = 4k + r, 0 \leq r \leq 3$ and $\pi = G(d,4) - \delta$. Then:*
$h^2(\mathcal{I}_S(k)) \geq \frac{2}{3}k^3 + k^2(\frac{r}{2} - 1) + k(\frac{7}{3} + \frac{r^2}{2} - 2r - \delta) - p_g$.

Proof. We have $\chi(\mathcal{I}_S(k)) = \chi(\mathcal{O}_{\mathbf{P}^4}(k)) - \chi(\mathcal{O}_S(k))$, so $h^2(\mathcal{I}_S(k)) = h^0(\mathcal{O}_{\mathbf{P}^4}(k)) - h^0(\mathcal{I}_S(k)) - \chi(\mathcal{O}_S(k)) + h^1(\mathcal{I}_S(k)) + h^3(\mathcal{I}_S(k))$, hence: $h^2(\mathcal{I}_S(k)) \geq h^0(\mathcal{O}_{\mathbf{P}^4}(k)) - h^0(\mathcal{I}_S(k)) - \chi(\mathcal{O}_S(k))$. Since S lies on an irreducible quartic hypersurface, and since $d > 16$, $h^0(\mathcal{I}_S(k)) = h^0(\mathcal{O}_{\mathbf{P}^4}(k-4))$. It follows that $h^2(\mathcal{I}_S(k)) \geq \frac{2}{3}k^3 + k^2 + \frac{7}{3}k + 1 - \chi(\mathcal{O}_S(k))$. By Riemann-Roch: $\chi(\mathcal{O}_S(k)) = \frac{kH.(kH-K)}{2} + \chi = \frac{d(k+1)k}{2} - k(\pi - 1) + 1 - q + p_g$. So $h^2(\mathcal{I}_S(k)) \geq$

$\frac{2}{3}k^3 + k^2 + \frac{7}{3}k - \frac{d(k+1)k}{2} + k(\pi - 1) - p_g$. Taking into account that: $d = 4k + r$ and that $\pi = G(d,4) - \delta = 1 + 2k^2 + kr + \frac{r}{2}(r-3) - \delta$, we get the result after a little computation. ∎

Corollary 7. *With notations as above:*
(i) If $p_g = 0$ then $h^2(\mathcal{I}_S(k)) \geq \rho_{\delta,r}(k)$ where:
$\rho_{\delta,r}(k) = \frac{2}{3}k^3 + k^2(\frac{r}{2} - 1) + k(\frac{7}{3} + \frac{r^2}{2} - 2r - \delta)$.
(ii) If $h^0(\omega_S(-1)) = 0$ and if the general hyperplane section of S is linearly normal in $\mathbf{P}^3$, then: $h^2(\mathcal{I}_S(k)) \geq \lambda_{\delta,r}(k)$, where:
$\lambda_{\delta,r}(k) = \frac{2}{3}k^3 + k^2(\frac{r}{2} - 3) + k(\frac{19}{3} + \frac{r^2}{2} - 3r - \delta) - \frac{r(r-5)}{2} - 4 + \delta$.
(iii) If $h^0(\omega_S(-1)) = 0$ then: $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,r}(k)$, where:
$\phi_{\delta,r}(k) = \frac{2}{3}k^3 + k^2(\frac{r}{2} - 3) + k(\frac{13}{3} + \frac{r^2}{2} - 3r - \delta) + 2r + \delta - 1 - \frac{r^2}{2}$.

Proof. (i) Follows directly from Lemma 6.
(ii) We use $p_g \leq \pi - d + 3$, see Lemma 5.
(iii) We use $p_g \leq \pi - \frac{d}{2}$. ∎

Remark 7.1. *For later use we observe that $\phi_{\delta,r}(k), \lambda_{\delta,r}(k), \rho_{\delta,r}(k)$ are increasing functions of k for $0 \leq \delta \leq 10$ and $0 \leq r \leq 3$.*

Lemma 8. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree d lying on a hypersurface of degree four with isolated singularities. Then $\pi = G(d,4) - \delta$ and, if $d = 4k + r, 0 \leq r \leq 3$, then: $\delta \leq 10$ if $r = 0$, $\delta \leq 9$ if $r = 1$ or $r = 3$; $\delta \leq 8$ if $r = 2$.*

Proof. This follows from the "Jacobi's formula": $\pi = 1 + \frac{d^2 - \mu}{8}$ (see [A], lemma 2.1). Since the hypersurface has only isolated singularities: $\mu \leq 81$, and since $G(d,4) = 1 + \frac{d^2 - 3r(4-r)}{8}$, we get $\delta = \frac{\mu - 3r(4-r)}{8}$ with $\mu \leq 81$ and we conclude. ∎

Arithmetically Cohen-Macaulay surfaces.

Lemma 9. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree d lying on an irreducible hypersurface of degree four. If S is arithmetically Cohen-Macaulay and if $h^0(\omega_S(-1)) = 0$ (resp. $p_g = 0$), then $d \leq 16$ (resp. $d \leq 12$).*

Proof. Since S is a.C.M., the minimal free resolution yields an exact sequence:

$$0 \to \bigoplus_{j=1}^{r} \mathcal{O}(-b_j) \to \bigoplus_{i=1}^{r+1} \mathcal{O}(-a_i) \to \mathcal{I}_S \to 0$$

Dualizing, we get:

$$0 \to \mathcal{O}(-6) \to \bigoplus_{i=1}^{r+1} \mathcal{O}(a_i - 6) \to \bigoplus_{j=1}^{r} \mathcal{O}(b_j - 6) \to \omega_S(-1) \to 0$$

If $h^0(\omega_S(-1)) = 0$ then $b^+ \leq 5$ (here $b^+ = max\{b_j\}$). Since $b^+ > a^+$, we get $a_i \leq 4, \forall i$. Since $r \geq 2$, $h^0(\mathcal{I}_S(4)) \geq 2$ and this implies, by Bezout, $d \leq 16$.

If $h^0(\omega_S) = 0$, the proof is similar. ∎

Remark 9.1. Thanks to the previous lemma, in the sequel, we will assume S not a.C.M., i.e. we will assume that the general hyperplane section, $C \subset \mathbf{P}^3$ of S is not projectively normal.

The case $d = 4k$.

Lemma 10. *Let $C \subset \mathbf{P}^3$ be a smooth, connected curve of degree $d = 4k$, $k \geq 4$ and genus π, lying on a smooth quartic surface. Assume $\pi = G(d, 4) - \delta$ with $\delta \leq 10$. Moreover suppose C non projectively normal. Then:*
(i) $g(\chi(C)) = G(d, 4) - 2$.
(ii) $\delta \geq 3$ and $c > e - 1$.
(iii) $k - 3 \leq e \leq k - 1$. Moreover if $e = k - 3$ then $\delta = 10$, $h^1(\mathcal{I}_C(k-2)) = 0, C$ is linearly normal and: $c \leq k + 3$ or $r_{k+1} = 2$.

Proof. (i) There are only two connected numerical characters of degree $4k$, length 4: $\phi = (k+3, k+2, k+1, k)$, $\chi = (k+2, k+2, k+1, k+1)$. If $\chi(C) = \phi$ then by [D] (see also [E]), then C is projectively normal, which is excluded. Hence $\chi(C) = \chi$. We have: $h_\phi(k) = 3, h_\phi(k+1) = 1$, $h_\chi(k) = 2, h_\chi(k+1) = 0$ while $h_\phi(t) = h_\chi(t)$ otherwise. It follows that $g(\chi) = g(\phi) - 2 = G(d, 4) - 2$.
(ii) Since C is not projectively normal, we have $g(C) < g(\chi(C))$, i.e. $\delta \geq 3$. Since $dk = 4k^2 > 2\pi - 2 = 4k^2 - 2\delta$, we get $e \leq k - 1$. Since C lies on an irreducible quartic surface and since $d > 16$, $h^0(\mathcal{I}_C(t)) = h^0(\mathcal{O}_{\mathbf{P}^3}(t-4))$ for $t \leq k$, and a simple computation shows that: $h^1(\mathcal{I}_C(k)) = \delta - 2 \neq 0$, hence $c \geq k$. It follows that $c > e - 1$.
(iii) We have seen (cf (ii)) that $e \leq k - 1$. Since C lies on an irreducible quartic surface, for $t \leq k, h^1(\mathcal{I}_C(t)) = dt - \pi + 1 + h^1(\mathcal{O}_C(t)) + h^0(\mathcal{O}_{\mathbf{P}^3}(t-4)) - h^0(\mathcal{O}_{\mathbf{P}^3}(t))$. So if $e < k - 2$, $h^1(\mathcal{I}_C(k-2)) = \delta - 10$, hence the only possibility is $\delta = 10$ and $h^1(\mathcal{I}_C(k-2)) = 0$, this implies also $e = k - 3$ (otherwise C would be projectively normal by Castelnuovo-Mumford's lemma). In this case, since $h^1(\mathcal{I}_C(k-2)) = 0$, by descending induction, $h^1(\mathcal{I}_C(t)) = 0$ if $t \leq k - 2$, in particular C is linearly normal.
If $\mathbf{I}(C)$ has a generator of degree $k + 1$, we can link C to a curve, X, by a complete intersection $(4, k+1)$. The curve X has degree 4 and $p_a(X) = -7$:

since X lies on a smooth quartic surface the only possibility is the disjoint union of two double lines (of arithmetic genus -3), see lemma below. In particular $h^1(\mathcal{I}_X(-t)) = 0$ if $t > 2$. By liaison this implies $c \leq k+3$.
If $\mathbf{I}(C)$ has no generator of degree $k+1$, then $r_{k+1} = 2$. ∎

Lemma 11. *Let X be a curve of degree 4 and arithmetic genus -7, lying on a smooth quartic surface. Then X is the disjoint union of two double lines of arithmetic genus -3.*

Proof. We have $X^2 = 2p-2 = -16$. Clearly X is non-reduced, so it must contain a line with multiplicity ≥ 2 or X is a double conic. Checking case by case we get the lemma. ∎

Proposition 12. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d = 4k, k \geq 5$ lying on a hyperquartic with isolated singularities.*
(i) If $h^0(\omega_S(-1)) = 0$, then $d \leq 24$.
(ii) If $p_g = 0$, then $d \leq 20$.

Proof. We have $\pi = G(d,4) - \delta$ and from Lemma 8, we may assume $\delta \leq 10$. From Lemma 10 (see Remark 9.1): $k-3 \leq e \leq k-1$ and $c > e-1$. First we make the following:

Claim: $h^2(\mathcal{I}_S(k)) \leq 6\delta - 12$ if $e \geq k-2$ and, if $e = k-3$ then C is linearly normal, $\delta = 10$, and $h^2(\mathcal{I}_S(k)) \leq 54$.

Assume this for a while and let's conclude the proof.
(i) If $e = k-3$, since C is linearly normal, we have (see Corollary 7 and Remark 7.1): $h^2(\mathcal{I}_S(k)) \geq \lambda_{\delta,0}(7) = 122 - 6\delta$, if $k \geq 7$. From the claim it follows that: $54 \geq 122 - 6\delta$, which is impossible if $\delta = 10$.
Now assume $e \geq k-2$. This time we use $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,0}(k)$. Arguing as above: $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,0}(7) = 111 - 6\delta$. Combining with the claim: $123 \leq 12\delta$, which is impossible for $\delta \leq 10$. It follows that it must be $k \leq 6$, i.e. $d \leq 24$.
(ii) We argue as before but using $h^2(\mathcal{I}_S(k)) \geq \rho_{\delta,0}(k) \geq \rho_{\delta,0}(6) = 122 - 6\delta$. If $e \geq k-2$, we get $122 - 6\delta \leq 6\delta - 12$ which is impossible for $\delta \leq 10$. If $e = k-3$, since $\delta = 10$, we have $h^2(\mathcal{I}_S(k)) \geq 62$ and we conclude with the claim.

To conclude, let's prove the claim:
We have $h^2(\mathcal{I}_S(k)) \leq (c-k).(g(\chi(C)) - \pi - \sum_{m \leq k+1} r_m)$, $(*)$, (see Remark 3.1).
Since $g(\chi(C)) = G(d,4) - 2$, it follows that: $h^2(\mathcal{I}_S(k)) \leq (c-k).(\delta - 2)$, $(**)$.
By Lemma 4: $c \leq d - 3e = 4k - 3e$, $(***)$.
If $e \geq k-2$, from $(**)$ and $(***)$ we get: $h^2(\mathcal{I}_S(k)) \leq 6\delta - 12$ and the claim is proved.

If $e = k-3$, by Lemma 10, $\delta = 10, h^1(\mathcal{I}_C(t)) = 0$ if $t \le k-2$, and $c \le k+3$ or $r_{k+1} = 2$.
- If $c \le k+3$, we get $h^2(\mathcal{I}_S(k)) \le 3\delta - 6 = 24$.
- In any case, from Lemma 4, $c \le k+9$. Since $r_{k+1} = 2$, from $(*)$, we get: $h^2(\mathcal{I}_S(k)) \le 9(\delta - 4) = 54$, since $\delta = 10$. ∎

The case $d = 4k+1$.

Lemma 13. *Let $C \subset \mathbf{P}^3$ be a smooth, connected curve of degree $d = 4k+1$, $k \ge 4$, and genus $\pi = G(d,4) - \delta = 2k^2 + k - \delta$. Assume C lies on an irreducible quartic surface and C not projectively normal. Then, if $\delta \le 9$:*
(i) $\delta \ge 2$ and $g(\chi(C)) = G(d,4) - 1$.
(ii) $k-2 \le e \le k$.
(iii) $c > e-1$.

Proof. (i) There are two possible connected numerical characters: the maximal one, $\phi = (k+3, k+2, k+1, k+1)$ and $\chi = (k+2, k+2, k+2, k+1)$. If $\chi(C) = \phi$ then C is projectively normal ([D], [E]), so we may assume $\chi(C) = \chi$. We have $h_\phi(k+1) = 1$, $h_\chi(k+1) = 0$, while $h_\phi(t) = h_\chi(t)$ if $t \ne k+1$. It follows that $g(\chi) = g(\phi) - 1 = G(d,4) - 1$. Since C is not projectively normal, it must be $g(C) < g(\chi(C))$, i.e. $\delta \ge 2$.
(ii) Since $d(k+1) > 2\pi - 2$, we have $e \le k$.
Assume $e < k-2$. Then $h^0(\mathcal{O}_C(k-2)) = d(k-2) - \pi + 1 = 2k^2 - 8k - 1 + \delta$. On the other hand $h^0(\mathcal{I}_C(k-2)) = h^0(\mathcal{O}_{\mathbf{P}^3}(k-6))$ since C is contained in an irreducible quartic surface. Now we must have:
$h^0(\mathcal{O}_{\mathbf{P}^3}(k-2)) - h^0(\mathcal{I}_C(k-2)) \le h^0(\mathcal{O}_C(k-2))$, which is equivalent to: $2k^2 - 8k + 10 \le 2k^2 - 8k - 1 + \delta$, but this is impossible if $\delta \le 9$.
(iii) We have $h^1(\mathcal{I}_C(k)) = h^0(\mathcal{O}_C(k)) + h^0(\mathcal{I}_C(k)) - h^0(\mathcal{O}_{\mathbf{P}^3}(k))$. We have $h^0(\mathcal{I}_C(k)) = h^0(\mathcal{O}_{\mathbf{P}^3}(k-4))$, we get $h^1(\mathcal{I}_C(k)) = \delta - 1 + h^1(\mathcal{O}_C(k))$, since $\delta \ge 2$ by (i), it follows that $c \ge k$ hence $c > e-1$. ∎

Proposition 14. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d = 4k+1$, with $k \ge 4$. Assume S lies on an irreducible hypersurface of degree four with isolated singularities.*
(i) If $p_g = 0$, then $d \le 21$.
(ii) If $h^0(\omega_S(-1)) = 0$, then $d \le 25$.

Proof. (i) We have $\pi = G(d,4) - \delta$ and, by Lemma 8, we may assume $\delta \le 9$. We may assume C not projectively normal (Remark 9.1). By Lemma 13, $k-2 \le e \le k$ and $c > e-1$. By Corollary 7 and Remark 7.1, we have $h^2(\mathcal{I}_S(k)) \ge \rho_{\delta,1}(6) = 131 - 6\delta$. On the other hand, by Corollary 3, $h^2(\mathcal{I}_S(k)) \le (c-k).(g(\chi(C)) - \pi)$. Since, by Lemma 4, $c \le k+7$ and since $g(\chi(C)) \le G(d,4) - 1$, by the previous lemma, it follows that $h^2(\mathcal{I}_S(k)) \le$

$7(\delta-1)$. Combining with the previous inequality yields a contradiction if $k \geq 6$.
(ii) We argue as above but using $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,1}(7) = 120 - \frac{1}{2} - 6\delta$. We get a contradiction if $k \geq 7$. ∎

The case $d = 4k+2$.

Lemma 15. *Let $C \subset \mathbf{P}^3$ be a smooth, connected curve of degree $d = 4k+2$ with $k \geq 4$ and genus $\pi = G(d,4) - \delta = 2k^2 + 2k - \delta$, lying on an irreducible quartic surface. Assume C non projectively normal, then, if $\delta \leq 8$:*
(i) $k-2 \leq e \leq k$ and $g(\chi(C)) \leq G(d,4)$.
(ii) $c > e-1$.
(iii) $c \leq k+8$.

Proof. (i) Since $d(k+1) > 2\pi - 2$, $e \leq k$.
Since C lies on an irreducible quartic surface $h^0(\mathcal{I}_C(t)) = h^0(\mathcal{O}_{\mathbf{P}^3}(t-4))$ for $t \leq k$. It follows that $h^1(\mathcal{I}_C(t)) = dt - \pi + 1 + h^1(\mathcal{O}_C(t)) + h^0(\mathcal{O}_{\mathbf{P}^3}(t-4)) - h^0(\mathcal{O}_{\mathbf{P}^3}(t))$. In particular $h^1(\mathcal{I}_C(k-2)) = \delta - 13 + h^1(\mathcal{O}_C(k-2))$. Since $\delta \leq 8$, this implies $e \geq k-2$.
Finally it is clear that $g(\chi(C)) \leq G(d,4)$ (by the way notice that in this case [D], [E] do not apply).
(ii) We have $h^1(\mathcal{I}_C(k)) = \delta - 1 + h^1(\mathcal{O}_C(k))$ and $h^1(\mathcal{I}_C(k-1)) = \delta - 5 + h^1(\mathcal{O}_C(k-1))$. If $h^1(\mathcal{I}_C(k)) \neq 0$, then $c > e-1$ and we are done. Assume $h^1(\mathcal{I}_C(k)) = 0$, then $\delta = 1$ and $e = k-1$. If $h^1(\mathcal{I}_C(k-1)) \neq 0$, $c > e-1$. Finally, if $h^1(\mathcal{I}_C(k-1)) = 0$, observe that, since for degree reasons the exact sequence:
$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C\cap H}(t) \to 0$ is exact on global sections for $t \leq k$, by descending induction we get $h^1(\mathcal{I}_C(t)) = 0$ for $t \leq k$. Since by assumption, C is not projectively normal, this implies $c > k$ and the condition $c > e-1$ is satisfied.
(iii) Follows from (i) and Lemma 4. ∎

Proposition 16. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d = 4k+2$, $k \geq 4$, lying on a quartic hypersurface with isolated singularities.*
(i) If $p_g = 0$, then $d \leq 22$.
(ii) If $h^0(\omega_S(-1)) = 0$, then $d \leq 26$.

Proof. (i) We have $\pi = G(d,4) - \delta$ and we may assume $\delta \leq 8$ (Lemma 8). We may assume C not projectively normal (Remark 9.1). By the previous lemma, $k-2 \leq e \leq k$ and $c > e-1$. By Corollary 7 and Remark 7.1, we have $h^2(\mathcal{I}_S(k)) \geq \rho_{\delta,2}(6) = 146 - 6\delta$. By Corollary 3, $h^2(\mathcal{I}_S(k)) \leq (c-k).(g(\chi(C)) - \pi)$. Since $c \leq k+8$ and $g(\chi(C)) \leq G(d,4)$ (Lemma 15), we get: $h^2(\mathcal{I}_S(k)) \leq 8\delta$. Combining everything, we get a contradiction if $k \geq 6$.

(ii) We argue as above, but using $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,2}(7) = 134 - 6\delta$. ■

The case $d = 4k + 3$.

Lemma 17. *Let $C \subset \mathbf{P}^3$ be a smooth, connected curve of degree $d = 4k+3, k \geq 3$ and genus $\pi = G(d,4) - \delta = 2k^2 + 3k + 1 - \delta$. Assume C lies on a irreducible quartic surface and that C is not projectively normal. Then, if $\delta \leq 9$:*
(i) $g(\chi(C)) = G(d,4) - 1$ and $\delta \geq 2$.
(ii) $k - 2 \leq e \leq k$.
(iii) $c \geq k + 1$, in particular $c > e - 1$.

Proof. (i) There are two possible connected numerical characters: the maximal one, $\phi = (k+3, k+3, k+2, k+1)$ and $\chi = (k+3, k+2, k+2, k+2)$. Since C is not projectively normal, by [D], [E], $\chi(C) = \chi$. We have $h_\phi(k+1) = 2, h_\chi(k+1) = 1$ and $h_\phi(t) = h_\chi(t)$ otherwise. This shows that $g(\chi) = g(\phi) - 1 = G(d,4) - 1$. Taking into account that $g(C) < g(\chi(C))$ because C is not projectively normal, we get $\delta \geq 2$.
(ii) Since $d(k+1) > 2\pi - 2$, $e \leq k$.
If $e < k - 2$ then $h^0(\mathcal{O}_C(k-2)) = d(k-2) - \pi + 1$. Since C is contained in a irreducible quartic surface, $h^0(\mathcal{I}_C(k-2)) = h^0(\mathcal{O}_{\mathbf{P}^3}(k-6))$, and we get $h^1(\mathcal{I}_C(k-2)) = \delta - 16$, which is absurd if $\delta \leq 9$.
(iii) We have $h^0(\mathcal{I}_C(k+1)) = h^0(\mathcal{O}_{\mathbf{P}^3}(k-3))$ (otherwise C would be linked to a line by a complete intersection $(4, k+1)$ and thus C would be projectively normal). It follows (since $e \leq k$, see (ii)) that $h^1(\mathcal{I}_C(k+1)) = \delta - 1 \neq 0$ (because $\delta \geq 2$, see (i)), hence $c \geq k+1$ and in particular $c > e - 1$. ■

Proposition 18. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d = 4k+3$ lying on a quartic hypersurface with isolated singularities.*
(i) If $p_g = 0$, then $d \leq 23$.
(ii) If $h^0(\omega_S(-1)) = 0$, then $d \leq 27$.

Proof. (i) By Lemma 8, $\pi = G(d,4) - \delta$ with $\delta \leq 9$. By Corollary 3 we have $h^2(\mathcal{I}_S(k)) \leq (c-k).(g(\chi(C) - \pi)$. We may assume C not projectively normal (Remark 9.1). Since $e \geq k-2$ by the previous lemma, by Lemma 4, we get $c \leq k+9$. Finally, since $g(\chi(C)) = G(d,4) - 1$, it follows that $h^2(\mathcal{I}_S(k)) \leq 9(\delta - 1)$ (*). On the other hand, by Corollary 7 and Remark 7.1, if $k \geq 6$, $h^2(\mathcal{I}_S(k)) \geq \rho_{\delta,3}(6) = 167 - 6\delta$. Combining with (*): $15\delta \geq 176$ which implies $\delta \geq 12$. So in our case, it must be $k \leq 5$ i.e. $d \leq 23$.
(ii)We argue as above but using $h^2(\mathcal{I}_S(k)) \geq \phi_{\delta,3}(7) = 155 - \frac{1}{2} - 6\delta$ if $k \geq 7$. (See Corollary 7) Combining with (*): $15\delta \geq 164 - \frac{1}{2}$, which implies $\delta \geq 11$. It follows that under our assumptions, we must have $k \leq 6$. ■

Conclusion.

Gathering everything together (Lemma 9, Prop. 12, 14, 16 and 18):

Theorem 19. *Let $S \subset \mathbf{P}^4$ be a smooth surface of degree $d = 4k + r$, $0 \leq r \leq 3$, lying on a quartic hypersurface with isolated singularities.*
(i) If $p_g = 0$, then $k \leq 5$ (in particular $d \leq 23$).
(ii) If $h^0(\omega_S(-1)) = 0$, then $k \leq 6$ (in particular $d \leq 27$).

Corollary 20. *Let $S \subset \mathbf{P}^4$ be a smooth surface lying on a quartic hypersurface with isolated singularities.*
(i) If S is not of general type, then $d \leq 27$.
(ii) If S is rational, then $d \leq 23$.

Proof. Just observe that if S is not of general type then $h^0(\omega_S(-1)) = 0$■

Another immediate consequence:

Corollary 21. *Let $V \subset \mathbf{P}^5$ be a smooth threefold of degree d, lying on a quartic hypersurface, Σ. Assume $h^0(\omega_V) = 0 = h^1(\omega_V(-1))$.*
If $dim(Sing(\Sigma)) \leq 1$, then $d \leq 27$.

Proof. Let S be a general hyperplane section of V. From the exact sequence:

$$0 \to \omega_V(-1) \to \omega_V \to \omega_S(-1) \to 0$$

we get $h^0(\omega_S(-1)) = 0$. Then, since S lies on an hyperquartic with isolated singularities, apply Theorem 19. ■

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